Effect of Initial Stresses on the Small Deformations of a Composite Rod

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A system of linear equations governing the small deformations of an initially stressed, curved, twisted composite rod is obtained through the use of the principle of virtual work. Apart from the extensional and bending deformations, transverse shear as well as warping of the rod cross section are incorporated into the assumed displacement field. The resulting equations are applicable for transversely isotropic rods for a wide range of values of the ratio of the shear modulus to Young's modulus. Special cases of the governing equations are also presented for plane-curved symmetric rods.

I. Introduction

IN this paper, we consider the effect of initial stresses on the small deformations of a curved, twisted rod. The rod is assumed to be transversely isotropic with a heterogeneous cross section. The effect of initial axial stress on the torsional rigidity of straight uniform rods is well-known.[1,2] An examination of the formula in Ref. 1 for the effective torsional rigidity reveals that the effect of axial stress is particularly significant whenever the shear modulus of the material is small compared to Young's modulus. Corresponding to this range of values of the shear modulus, the shear deformation of the rod may be significant. In the following treatment, we obtain the effective torsional rigidity for space curved rods in the presence of shear deformation as part of our results. The analysis of the initial stress problem is based on the linearized three-dimensional initial stress problem, and the associated variational principles discussed in Ref. 2. In conjunction with the principle of virtual work, we assume a displacement field incorporating transverse shear deformation and warping of the cross section. In the subsequent considerations, approximations are introduced on the basis of the thinness of the rod to simplify the resulting relations. Inertia forces, associated with the small deformations, are included through the use of D'Alembert's principle.

A system of equations for the treatment of symmetric, plane-curved rods is obtained by specializing the general relations. These equations readily uncouple into two sets—one describing the in-plane deformations, the other describing the out-of-plane deformations of the rod. Explicit forms of these relations are also stated for the case of negligible transverse shear deformation.

An analysis of superposed small displacements on finite deformations of rods has been previously given by Green et al.[3] The constitutive relations obtained in Ref. 3, on the basis of thermodynamically consistent strain energy functions, introduce unnecessary complexity in an approximate treatment of practical rod problems, such as the dynamic stability of underwater cables and the stability of thin rods. Although it is possible to obtain the equations for superposed small deformations of an initially stressed rod by perturbing the results obtained for large deformations of an initially unstressed rod, most of the large deformation theories[4-8] reveal features that render this procedure impractical for technical applications.[9] Some of these features are the presence of generalized constitutive relations[4,5] and the absence of constitutive relations.[6,7] The approximate derivation presented here appears to be more direct for practical applications.

Received Aug. 22, 1977; revision received Nov. 14, 1977.

Index categories: Structural Composite Materials; Marine Mooring Systems and Cable Mechanics.

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II. Initial Equilibrium Configuration

In portions of the subsequent analysis, indicial notations[10,11] will be used. Latin indices range over 1, 2, 3 and Greek indices range over 1, 2. The summation convention holds in each case.

Consider a rod in static equilibrium under initial stresses. Letting C represent the locus of material particles at similar locations, a particle on C has the position vector

$$\overset{(o)}{r}=\overset{(o)}{r}(s) \tag{1}$$

Let A_i represent a unit orthogonal triad with its origin on C such that

$$A_3=\partial \overset{(o)}{r}/\partial s \qquad A_i\cdot A_j=\delta_{ij} \tag{2}$$

The spacial derivatives of A_i may be written as

$$\partial A_i/\partial s=-e_{ijk}\mathcal{K}_j A_k \tag{3}$$

where $\mathcal{K}_j$ are some measures of the curvatures of the space curve C, and e_{ijk} is the permutation symbol.

The position vector $\overset{(o)}{R}$ of a general particle in the initial configuration of the rod is assumed to be represented in the form

$$\overset{(o)}{R}=\overset{(o)}{r}+\theta_\alpha A_\alpha+\mathcal{K}_3\Phi(\theta_1,\theta_2)A_3 \tag{4}$$

where θ_i represents a system of curvilinear coordinates with $\theta_3\equiv s$ and Φ represents the warping function for the cross section in accordance with the St. Venant theory of torsion. Although the presence of the Φ terms in Eq. (4) has no effect on the subsequent results, we introduce the term with the objective of obtaining the deformed position of the particle in agreement with Eq. (4) in form. Excepting the Φ term, the θ_α coordinate would be rectilinear.

The base vectors $g_i = \partial \overset{(o)}{R} / \partial \theta_i$ of the initial configuration are obtained from Eq. (4) in the form

$$g_\alpha = A_\alpha + \mathcal{K}_3 \Phi_{,\alpha} A_3, \quad g_3 = A_3 + \theta_\alpha A'_\alpha + (\mathcal{K}_3 A_3)' \Phi \tag{5}$$

where primes represent differentiations with respect to s.

We assume that

$$\max(\theta_\alpha) = b \qquad \max(\mathcal{K}_i) = 1/R \qquad b/R \ll 1$$

$$\Phi = O(b^2) \qquad f_{,\alpha} = O(f/b) \qquad f' = O(f/R) \tag{6}$$

From Eqs. (5) and (6), it follows that

$$\sqrt{g} \equiv g_1 \times g_2 \cdot g_3 \doteq 1 + \theta_2 \mathcal{K}_1 - \theta_1 \mathcal{K}_2 \tag{7}$$

Insofar as the initial stresses are concerned, we assume that

$$(\sqrt{g}\, \overset{(o)}{\sigma}{}^i)_{,i} + \sqrt{g}\, \overset{(o)}{f} = 0 \tag{8}$$

inside the rod

$$\overset{(o)}{\sigma}{}^i (g_i \cdot \nu) = \overset{(o)}{t}{}^\nu \tag{9a}$$

on the lateral rod surface S and

$$\overset{(o)}{\sigma}{}^3 = \overset{(o)}{t}{}^3 (\theta_1, \theta_2, s_\alpha) \tag{9b}$$

on the bounding cross section, s_α = const, of the rod. In Eqs. (8) and (9), $\overset{(o)}{\sigma}{}^i$ represents the stress vector acting at a point on the surface θ_i = const, and $\overset{(o)}{f}$, $\overset{(o)}{t}{}^\nu$, and $\overset{(o)}{t}{}^3$ represent the body force, traction vector on the lateral surface, and traction vectors on the end surfaces, respectively. Here ν represents the outward normal to the lateral surface. We assume that the initial stress distribution in the rod is known from calculations, such as, for example, that given in Ref. 9.

III. Principle of Virtual Work

We now consider infinitesimal deformations of the rod from the initial equilibrium. After deformation, a particle at $\overset{(o)}{R}$ has moved to R,

$$R = \overset{(o)}{R} + U \tag{10}$$

where U is the displacement vector.

The components of Green's strain are given by

$$2e_{ij} = g_i \cdot U_{,j} + g_j \cdot U_{,i} + U_{,i} \cdot U_{,j} \tag{11}$$

In the following we employ the principle of virtual work in the form

$$\iiint \sqrt{g} \{ (\overset{(o)}{\sigma}{}^{ij} + \sigma^{ij}) \delta e_{ij} - (\overset{(o)}{f} + f) \cdot \delta U \} d\theta_1 d\theta_2 d\theta_3$$

$$- \iint (\overset{(o)}{t}{}^\nu + t^\nu) \cdot \delta U dS - \{ \iint \sqrt{g} (\overset{(o)}{t}{}^3 + t^3) \cdot \delta U d\theta_1 d\theta_2 \}_{s_1}^{s_2} = 0 \tag{12}$$

Here, $\overset{(o)}{\sigma}{}^{ij}$ are the components of $\overset{(o)}{\sigma}{}^i$ with respect to g_j,

$$\overset{(o)}{\sigma}{}^i = \overset{(o)}{\sigma}{}^{ij} g_j \tag{13}$$

σ^{ij} are the incremental Kirchhoff stress components expressed in terms of e_{ij} through appropriate constitutive relations, and f, t^ν, and t^3 are incremental body force, traction vector on the lateral surface, and traction vector on the end surfaces, respectively.

For transversely isotropic materials, we assume constitutive relations of the form

$$e_{\alpha\beta} = \frac{1+\nu^*}{E^*} \sigma^{\alpha\beta} - \frac{\nu^*}{E^*} \sigma^{\gamma\gamma} \delta_{\alpha\beta} - \frac{\nu}{E} \sigma^{33} \delta_{\alpha\beta}$$

$$e_{3\alpha} = \frac{\sigma^{3\alpha}}{2G}, \quad e_{33} = \frac{\sigma^{33} - \nu \sigma^{\gamma\gamma}}{E} \tag{14}$$

where E^* and ν^* represent Young's modulus and Poisson's ratio in the surface of isotropy (i.e., the surface θ_3 = const.), E, ν represent the corresponding constants in g_3 direction, and G represents the shear modulus. We note that G is an independent elastic constant in the case of transverse isotropy.

We next assume that the displacement distribution U is of the form

$$U = u + \omega \times \theta + (\omega' \cdot A_3) \Phi A_3 \qquad \theta \equiv \theta_\alpha A_\alpha \tag{15}$$

where $u = u(s)$ is the translational and $\omega = \omega(s)$ the rotational displacements, and where the last term represents the warping of the cross section. The following order of magnitude relation between the infinitesimal quantities u and ω are assumed to hold

$$|\omega \times \theta| = O(|u| b/R) \tag{16}$$

In Eq. (15), Φ is the warping function introduced in Eq. (4). For a homogeneous cross section A, Φ satisfies Laplace's equation

$$\Phi_{,\alpha\alpha} = 0 \tag{17}$$

in A, and

$$\Phi_{,\alpha} \nu_\alpha = \theta_2 \nu_1 - \theta_1 \nu_2 \tag{18}$$

on the boundary Γ of the region A, where $\nu = \nu_\alpha A_\alpha$ is the outward normal to Γ. For a rod with a heterogeneous cross section, let A be divided into N regions $A^{(n)}$, $n = 1,2,\ldots N$. Within each region, let the elastic constants be homogeneous. Then an extension of St. Venant free torsion solution to composite cross sections reveals that 1) $\Phi = \Phi^{(n)}$ in $A^{(n)}$ where $\Phi^{(n)}$ satisfies Eq. (17); 2) Φ is single-valued and continuous throughout A; 3) $\Phi^{(n)}$ satisfies Eq. (18) on Γ; and 4) $\Phi^{(n)}$ satisfies the jump conditions

$$G^{(n)} \Phi_{,\alpha}^{(n)} \nu_\alpha^{(n)} - G^{(m)} \Phi_{,\alpha}^{(m)} \nu_\alpha^{(n)} = [G^{(n)} - G^{(m)}]$$

$$\times [\theta_2 \nu_1^{(n)} - \theta_1 \nu_2^{(n)}] \tag{19}$$

on the common boundary separating $A^{(n)}$ and $A^{(m)}$. In Eq. (19), $G^{(i)}$ represents the shear modulus in $A^{(i)}$ with $G=0$ for void regions and $\nu^{(n)}$ is the outward normal to the boundary $\Gamma^{(n)}$ of $A^{(n)}$. No summation on n or m is implied.

Use of Eq. (15) in Eq. (10) shows that the deformed configuration of the rod can be expressed as

$$R = r^* + \theta_\alpha A^*_\alpha + \mathcal{K}^*_3 \Phi A^*_3 \tag{20}$$

where

$$r^* = \overset{(o)}{r} + u \tag{21}$$

$$A^*_1 = A_1 + \omega_3 A_2 - \omega_2 A_3 \tag{22a}$$

$$A^*_2 = A_2 + \omega_1 A_3 - \omega_3 A_1 \tag{22b}$$

$$A^*_3 = A_3 + \omega_2 A_1 - \omega_1 A_2 \tag{22c}$$

$$\mathcal{K}^*_3 = \mathcal{K}_3 + \omega' \cdot A_3 \tag{22d}$$

In Eq. (20), certain higher order terms have been omitted.

Noting that the linear terms in $U,_i$ in δe_{ij} can be omitted in expressing $\sigma^{ij}\,\delta e_{ij}$, we write Eq. (12) in the form

$$\iiint\sqrt{g}\{\tfrac{1}{2}(\overset{(o)}{\sigma}{}^{ij}+\sigma^{ij})(g_i\cdot\delta U,_j+g_j\cdot\delta U,_i)$$
$$+\tfrac{1}{2}\overset{(o)}{\sigma}{}^{ij}(U,_i\cdot\delta U,_j+U,_j\cdot\delta U,_i)-(\overset{(o)}{f}+f)\cdot\delta U\}d\theta_1 d\theta_2 d\theta_3$$
$$-\iint(\overset{(o)}{t}{}^{\nu}+t^{\nu})\cdot\delta U dS-\{\iint\sqrt{g}(\overset{(o)}{t}{}^3+t^3)\cdot U d\theta_1 d\theta_2\}_{s_1}^{s_2}=0 \quad (23)$$

Appropriate integration by parts of term $\overset{(o)}{\sigma}{}^{ij}(g_i\cdot\delta U,_j+g_j\cdot\delta U,_i)$ and use of the equilibrium relation [Eq. (8)] and boundary condition [Eq. (9a)], reduces Eq. (23) to the form

$$\iiint\sqrt{g}\{\tfrac{1}{2}\sigma^{ij}(g_i\cdot\delta U,_j+g_j\cdot\delta U,_i)+\tfrac{1}{2}\overset{(o)}{\sigma}{}^{ij}(U,_i\cdot\delta U_j+U,_j\cdot\delta U,_i)$$
$$-f\cdot\delta U\}d\theta_1 d\theta_2 d\theta_3-\iint t^{\nu}\cdot\delta U dS-\{\iint\sqrt{g}t^3\cdot\delta U d\theta_1 d\theta_2\}_{s_1}^{s_2}=0 \quad (24)$$

We next assume that the variational quantities may be approximated as follows:

$$\delta U=\delta u+\delta\omega\times\theta$$
$$\delta U'=\delta u'+(\delta\omega\times\theta)'$$
$$\delta U,_\alpha=\delta\omega\times A_\alpha \quad (25)$$

Furthermore, taking $E^*\ll E$ in Eq. (14), Eq. (24) can be approximated by

$$\iiint\sqrt{g}\{Ee_{33}g_3\cdot\delta U'+2Ge_{3\alpha}(g_3\cdot\delta U,_\alpha+g_\alpha\cdot\delta U')+\overset{(o)}{\sigma}{}^{33}U'$$
$$\cdot\delta U'+\overset{(o)}{\sigma}{}^{3\alpha}(U'\cdot\delta U,_\alpha+U,_\alpha\cdot\delta U')-f\cdot\delta U\}d\theta_1 d\theta_2 d\theta_3$$
$$-\iint t^{\nu}\cdot\delta U dS-\{\iint\sqrt{g}t^3\cdot\delta U d\theta_1 d\theta_2\}_{s_1}^{s_2}=0 \quad (26)$$

where, in accordance with Eq. (25), the last term may be expressed as

$$\iint\sqrt{g}t^3\cdot\delta U d\theta_1 d\theta_2=N\cdot\delta u+M\cdot\delta\omega \quad (27)$$

where

$$N=\iint\sqrt{g}t^3 d\theta_1 d\theta_2,\quad M=\iint\sqrt{g}\theta\times t^3 d\theta_1 d\theta_2 \quad (28)$$

Expressing the lateral surface area element $dS=d\Gamma d\theta_3$ and

$$f=f^{(S)}+f^{(I)} \quad (29)$$

where $f^{(S)}$ is the static body force per unit volume and $f^{(I)}$ is the inertia force per unit volume in accordance with D'Alembert's principle, we have

$$\iint\sqrt{g}f\cdot\delta U d\theta_1 d\theta_2+\int\sqrt{g}t^{\nu}\cdot\delta U d\Gamma=p\cdot\delta u+q\cdot\delta\omega \quad (30)$$

In Eq. (30)

$$p=p^{(S)}-p^{(I)}\qquad q=q^{(S)}-q^{(I)} \quad (31)$$

where

$$p^{(S)}=\int\sqrt{g}t^{\nu}d\Gamma+\iint\sqrt{g}f^{(S)}d\theta_1 d\theta_2$$
$$p^{(I)}=\iint\sqrt{g}\rho\ddot{U}d\theta_1 d\theta_2$$
$$q^{(S)}=\int\sqrt{g}\theta\times t^{\nu}d\Gamma+\iint\sqrt{g}\theta\times f^{(S)}d\theta_1 d\theta_2$$
$$q^{(I)}=\iint\sqrt{g}\theta\times\rho\ddot{U}d\theta_1 d\theta_2 \quad (32)$$

with ρ representing the density of the constituent material at $(\theta_1,\theta_2,\theta_3)$.

Using Eqs. (27) and (30) in Eq. (26), we write

$$\iiint\sqrt{g}\{[Ee_{33}g_3+2Ge_{3\alpha}g_\alpha+\overset{(o)}{\sigma}{}^{33}U'+\overset{(o)}{\sigma}{}^{3\alpha}U,_\alpha]\cdot\delta U'$$
$$+[2Ge_{3\alpha}g_3+\overset{(o)}{\sigma}{}^{3\alpha}U']\cdot\delta U,_\alpha\}d\theta_1 d\theta_2 d\theta_3$$
$$-\int\{p\cdot\delta u+q\cdot\delta\omega\}d\theta_3-[N\cdot\delta u+M\cdot\delta\omega]_{s_1}^{s_2}=0 \quad (33)$$

Integrating the $\delta U'$ term by parts, we next obtain Eq. (33) in the form

$$\{\iint\sqrt{g}\tau^3\cdot\delta U d\theta_1 d\theta_2-N\cdot\delta u-M\cdot\delta\omega\}_{s_1}^{s_2}-\iiint\{(\sqrt{g}\tau^3)'$$
$$\cdot\delta U-\tau^\alpha\cdot U,_\alpha\}d\theta_1 d\theta_2 d\theta_3-\int\{p\cdot\delta u+q\cdot\delta\omega\}d\theta_3=0 \quad (34)$$

where

$$\tau^3=Ee_{33}g_3+2Ge_{3\alpha}g_\alpha+\overset{(o)}{\sigma}{}^{33}U'+\overset{(o)}{\sigma}{}^{3\alpha}U,_\alpha$$
$$\tau^\alpha=2Ge_{3\alpha}g_3+\overset{(o)}{\sigma}{}^{3\alpha}U' \quad (35)$$

IV. Constitutive Relations

The boundary terms in Eq. (34) give

$$N=\iint\sqrt{g}\tau^3 d\theta_1 d\theta_2,\quad M=\iint\sqrt{g}\theta\times\tau^3 d\theta_1 d\theta_2 \quad (36)$$

for an arbitrary choice of the end points s_1 and s_2.

Consistent with earlier approximations, in τ^3 we may express e_{33} and $e_{3\alpha}$ in the form

$$e_{33}=g_3\cdot\{u'+(\omega\times\theta)'\}$$
$$e_{3\alpha}=A_\alpha\cdot\{u'+(\omega\times\theta)'\}+g_3\cdot\omega\times A_\alpha+(\omega'\cdot A_3)\Phi,_\alpha \quad (37)$$

Furthermore, $\sqrt{g}$, g_3, and g_α may also be approximated by

$$\sqrt{g}\approx 1,\quad g_3=A_3+\theta'\quad g_\alpha=A_\alpha \quad (38)$$

A scalar representation of Eq. (36) follows upon expanding τ^3 in the form

$$\tau^3=\tau^{3i}A_i \quad (39)$$

where, with the use of Eqs. (37), (38), and (35), we have

$$\tau^{31}=G\{\gamma_1+\kappa_3(\Phi,_1-\theta_2)\}-E\mathcal{K}_3\theta_2(\epsilon+\kappa_1\theta_2-\kappa_2\theta_1)$$
$$+\overset{(o)}{\sigma}{}^{33}(\gamma_1-\kappa_3\theta_2)+\overset{(o)}{\sigma}{}^{33}\omega_2-\overset{(o)}{\sigma}{}^{32}\omega_3+\eta_1(\theta_1,\theta_2,\theta_3) \quad (40a)$$

$$\tau^{32}=G\{\gamma_2+\kappa_3(\Phi,_2+\theta_1)\}+E\mathcal{K}_3\theta_1(\epsilon+\kappa_1\theta_2-\kappa_2\theta_1)$$
$$+\overset{(o)}{\sigma}{}^{33}(\gamma_2+\kappa_3\theta_1)+\overset{(o)}{\sigma}{}^{31}\omega_3-\overset{(o)}{\sigma}{}^{33}\omega_1+\eta_2(\theta_1,\theta_2,\theta_3) \quad (40b)$$

$$\tau^{33}=E(\epsilon+\kappa_1\theta_2-\kappa_2\theta_1)-E\mathcal{K}_3\{\theta_2(\gamma_1-\kappa_3\theta_2)$$
$$-\theta_1(\gamma_2+\kappa_3\theta_1)\}+\overset{(o)}{\sigma}{}^{33}(\epsilon+\kappa_1\theta_2-\kappa_2\theta_1)+\overset{(o)}{\sigma}{}^{32}\omega_1-\overset{(o)}{\sigma}{}^{31}\omega_2 \quad (40c)$$

where

$$\gamma_1=u_1'-\omega_2-\mathcal{K}_3u_2+\mathcal{K}_2u_3,\quad \gamma_2=u_2'+\omega_1-\mathcal{K}_1u_3+\mathcal{K}_3u_1$$
$$\epsilon=u_3'-\mathcal{K}_2u_1+\mathcal{K}_1u_2,\quad \kappa_3=\omega_3'-\mathcal{K}_2\omega_1+\mathcal{K}_1\omega_2$$
$$\kappa_1=\omega_1'+\mathcal{K}_2\omega_3,\quad \kappa_2=\omega_2'-\mathcal{K}_1\omega_3 \quad (41)$$

In Eqs. (40), we have retained the $E\mathcal{K}_3$ terms in $\tau^{3\alpha}$. In the event G is of the order of Eb/R, these terms may be significant. To maintain the symmetry of the expressions, we also retain the small-order terms containing $E\mathcal{K}_3$ in τ^{33}.

The functions η_α appearing in the expressions for $\tau^{3\alpha}$ need special consideration. The displacement field, assumed in conjunction with the variational principle, leads to transverse shear stresses of the form $G\gamma_\alpha$. These stresses along with τ^{33} do not satisfy the three-dimensional equilibrium equations. The functions η_α are added to the expressions for $\tau^{3\alpha}$, obtained from Eqs. (37-39) in an attempt to satisfy the three-dimensional equilibrium equations. However, it is not possible to choose η_α to satisfy these equations in the general case. In what follows, η_α are assumed to be taken in such a way as to satisfy the equilibrium equations exactly in the case of in-plane bending of straight untwisted rods. We note that this choice of η_α leads to an estimate of the shear center location for the rod cross section in accordance with the standard cantilever case.[12] The details regarding the choice of η_α and its consequences on the constitutive relations may be obtained from Ref. 12.

With the aid of Eqs. (40), we have in the component representations of N and M,

$$N = N_i A_i \qquad M = M_i A_i \tag{42}$$

$$\begin{aligned}
N_1 &= N_1^* - \overset{(o)}{N}_2 \omega_3 + \overset{(o)}{N}_3 \omega_2, \quad M_1 = M_1^* - P_{21}^{(o)} \omega_2 + P_{22}^{(o)} \omega_1 \\
N_2 &= N_2^* - \overset{(o)}{N}_3 \omega_1 + \overset{(o)}{N}_1 \omega_3, \quad M_2 = M_2^* - P_{12}^{(o)} \omega_1 + P_{11}^{(o)} \omega_2 \\
N_3 &= N_3^* - \overset{(o)}{N}_1 \omega_2 + \overset{(o)}{N}_2 \omega_1, \quad M_3 = M_3^* - \overset{(o)}{M}_1 \omega_2 + \overset{(o)}{M}_2 \omega_1 + P_{\alpha\alpha}^{(o)} \omega_3
\end{aligned} \tag{43}$$

$$\begin{aligned}
N_1^* &= (\overline{GA} + \overset{(o)}{N}_3)\gamma_1 - \mathcal{K}_3\{EA\,\theta_2^{(n)}\epsilon + EI_{22}\kappa_1 - \overline{EI}_{12}\kappa_2\} - \overset{(o)}{M}_1\kappa_3 \\
N_2^* &= (\overline{GA} + \overset{(o)}{N}_3)\gamma_2 + \mathcal{K}_3\{EA\theta_1^{(n)}\epsilon + \overline{EI}_{12}\kappa_1 - \overline{EI}_{11}\kappa_2\} - \overset{(o)}{M}_2\kappa_3 \\
N_3^* &= (\overline{EA} + \overset{(o)}{N}_3)\epsilon - \mathcal{K}_3 EA[\theta_2^{(n)}\gamma_1 - \theta_1^{(n)}\gamma_2] \\
&\quad + (EA\theta_2^{(n)} + \overset{(o)}{M}_1)\kappa_1 - (\overline{EA}\theta_1^{(n)} - \overset{(o)}{M}_2)\kappa_2 + \mathcal{K}_3\overline{EI}_{\alpha\alpha}\kappa_3
\end{aligned} \tag{44}$$

$$\begin{aligned}
M_1^* &= (\overline{EI}_{22} + \overset{(o)}{M}_{22})\kappa_1 - (EI_{12} + \overset{(o)}{M}_{12})\kappa_2 + \mathcal{K}_3\overline{EI}_{\alpha\alpha 2}\kappa_3 \\
&\quad + (\overline{EA}\theta_2^{(n)} + \overset{(o)}{M}_1)\epsilon - \mathcal{K}_3\{\overline{EI}_{22}\gamma_1 - \overline{EI}_{12}\gamma_2\} \\
M_2^* &= (EI_{11} + \overset{(o)}{M}_{11})\kappa_2 - (\overline{EI}_{12} + \overset{(o)}{M}_{12})\kappa_1 - \mathcal{K}_3\overline{EI}_{\alpha\alpha 1}\kappa_3 \\
&\quad - (\overline{EA}\theta_1^{(n)} - \overset{(o)}{M}_2)\epsilon + \mathcal{K}_3\{\overline{EI}_{12}\gamma_1 - \overline{EI}_{11}\gamma_2\} \\
M_3^* &= (\overline{GJ} + \overset{(o)}{M}_{\alpha\alpha})\kappa_3 + \mathcal{K}_3\{\overline{EI}_{\alpha\alpha 2}\kappa_1 - \overline{EI}_{\alpha\alpha 1}\kappa_2\} \\
&\quad + \mathcal{K}_3\overline{EI}_{\alpha\alpha}\epsilon + (\overline{GA}\theta_1^{(s)} - \overset{(o)}{M}_2)\gamma_2 - (\overline{GA}\theta_2^{(s)} + \overset{(o)}{M}_1)\gamma_1
\end{aligned} \tag{45}$$

where

$$\begin{aligned}
&\overset{(o)}{N}_i = \iint \overset{(o)}{\sigma}{}^{3i}\,d\theta_1 d\theta_2, \qquad \overset{(o)}{P}_{\alpha\beta} = \iint \theta_\alpha \overset{(o)}{\sigma}{}^{3\beta}\,d\theta_1 d\theta_2 \\
&\overset{(o)}{M}_1 = \iint \theta_2 \overset{(o)}{\sigma}{}^{33}\,d\theta_1 d\theta_2, \quad \overset{(o)}{M}_2 = -\iint \theta_1 \overset{(o)}{\sigma}{}^{33}\,d\theta_1 d\theta_2 \\
&\overset{(o)}{M}_{\alpha\beta} = \iint \overset{(o)}{\sigma}{}^{33}\theta_\alpha\theta_\beta\,d\theta_1 d\theta_2 \\
&\overline{EA} = \iint E\,d\theta_1 d\theta_2, \quad \overline{GA} = \iint G\,d\theta_1 d\theta_2 \\
&\theta_\alpha^{(n)} = \iint E\theta_\alpha\,d\theta_1 d\theta_2/\overline{EA} \\
&\overline{EI}_{\alpha\beta} = \iint E\theta_\alpha\theta_\beta\,d\theta_1 d\theta_2, \quad \overline{EI}_{\alpha\beta\gamma} = \iint E\theta_\alpha\theta_\beta\theta_\gamma\,d\theta_1 d\theta_2 \\
&\overline{GJ} = \iint G[\theta_1\Phi_{,2} - \theta_2\Phi_{,1} + \theta_1^2 + \theta_2^2]\,d\theta_1 d\theta_2
\end{aligned} \tag{46}$$

The quantities $\theta_\alpha^{(s)}$ represent the shear center location for the rod cross section, in accordance with the standard cantilever case.[12]

We note that in integrating $\tau^{3\alpha}$ using Eq. (40), we have utilized the relation

$$\iint G(\Phi_{,1} - \theta_2)\,d\theta_1 d\theta_2 = \iint G(\Phi_{,2} + \theta_1)\,d\theta_1 d\theta_2 = 0 \tag{47}$$

which is implied by Eqs. (17-19).

We also note that the stress resultant N and the moment M, acting on the surface $\theta_3 = \text{const}$, are decomposed into two parts—one part due to the incremental strains γ_α, κ_α, κ_3, and ϵ; the other part due to the infinitesimal rotations of the initial stress resultants and moments. The constitutive relations for M_3^* involve the effective torsional rigidity exactly in the same form as in Refs. 1 and 2. Whenever $G = O(Eb/R)$, the transverse shear resultants N_α^* and the torsional moment M_3^* depend on the axial strain ϵ and on the incremental curvatures κ_α. Moreover, the transverse shear rigidity now depends on the initial axial stress $\overset{(o)}{\sigma}{}^{33}$. Furthermore, if the axis C is chosen in such a way that $\theta_1^{(s)} = \theta_2^{(s)} = 0$, the system of Eqs. (44) becomes symmetric. Considerable simplification of Eqs. (45) is possible whenever the rod possesses material as well as geometric symmetry about one or both of the coordinate axes θ_α.

V. Equilibrium Equations

Use of Eqs. (25) in Eq. (34) gives, as the Euler equations of the variational problem,

$$\iint (\sqrt{g}\,\tau^3)'\,d\theta_1 d\theta_2 + p = 0 \tag{48}$$

$$\iint \{\theta \times (\sqrt{g}\,\tau^3)' - A_\alpha \times \tau^\alpha\}\,d\theta_1 d\theta_2 + q = 0 \tag{49}$$

Equation (48), in conjunction with Eq. (36), gives

$$N' + p = 0 \tag{50}$$

In terms of the scalar components, Eq. (50) can be expressed as

$$\begin{aligned}
N_1' - N_2\mathcal{K}_3 + N_3\mathcal{K}_2 + p_1 &= 0 \\
N_2' - N_3\mathcal{K}_1 + N_1\mathcal{K}_3 + p_2 &= 0 \\
N_3' - N_1\mathcal{K}_2 + N_2\mathcal{K}_1 + p_3 &= 0
\end{aligned} \tag{51}$$

Equation (49) may be expressed in the form

$$M' + q - \iint \{\theta' \times \tau^3 + A_\alpha \times \tau^\alpha\}\,d\theta_1 d\theta_2 = 0 \tag{52}$$

From Eq. (35), we have

$$\begin{aligned}
A_\alpha \times \tau^\alpha &= \overset{(o)}{\sigma}{}^{3\alpha} A_\alpha \times U' - 2Ge_{3\alpha}g_3 \times A_\alpha \\
&= \overset{(o)}{\sigma}{}^{3\alpha} A_\alpha \times U' - g_3 \times [\tau^3 - Ee_{33}g_3 - \overset{(o)}{\sigma}{}^{33} U' - \overset{(o)}{\sigma}{}^{3\alpha} U_{,\alpha}]
\end{aligned} \tag{53}$$

where g_α has been approximated by A_α.

Using Eq. (53), and the relations $g_3 = A_3 + \theta'$ in Eq. (52), we write

$$\begin{aligned}
&M' + A_3 \times N - \iint \{\overset{(o)}{\sigma}{}^{33} A_3 \times U' \\
&\quad + \overset{(o)}{\sigma}{}^{3\alpha}(A_3 \times U_{,\alpha} + A_\alpha \times U')\}\,d\theta_1 d\theta_2 + q = 0
\end{aligned} \tag{54}$$

In terms of the scalar components, the vector equation (54) is equivalent to the system of equations

$$\begin{aligned}
&M_1' - M_2\mathcal{K}_3 + M_3\mathcal{K}_2 - N_2 + \overset{(o)}{N}_3(\gamma_2 - \omega_1) - \overset{(o)}{N}_2\epsilon + \overset{(o)}{N}_1\omega_3 \\
&\quad - \overset{(o)}{P}_{22}\kappa_1 + \overset{(o)}{P}_{12}\kappa_2 - \overset{(o)}{M}_2\kappa_3 + q_1 = 0 \\
&M_2' - M_3\mathcal{K}_1 + M_1\mathcal{K}_3 + N_1 - \overset{(o)}{N}_3(\gamma_1 + \omega_2) + \overset{(o)}{N}_1\epsilon - \overset{(o)}{N}_2\omega_3 \\
&\quad + \overset{(o)}{P}_{21}\kappa_1 - \overset{(o)}{P}_{11}\kappa_2 + \overset{(o)}{M}_1\kappa_3 + q_2 = 0
\end{aligned}$$

$$M_3' - M_1\mathcal{K}_2 + M_2\mathcal{K}_1 - \overset{(o)}{N}_1(\gamma_2 - \omega_1) + \overset{(o)}{N}_2(\gamma_1 + \omega_2) + \overset{(o)}{P}_{\alpha\alpha}\kappa_3 + q_3 = 0 \tag{55}$$

Equations (55, 51, 45, 43, and 41) constitute 24 equations for the 24 unknown quantities ϵ, κ_3, γ_α, κ_α, ω_i, κ_i, N_i, M_i, N_i^* and M_i^*. Insofar as particular problems are concerned, order of magnitude relations among the initial stress measured $\overset{(o)}{N}_i$, $\overset{(o)}{M}_i$, $\overset{(o)}{M}_{\alpha\beta}$, and $\overset{(o)}{P}_{\alpha\beta}$ may be introduced to justify further simplifications of the constitutive equations as well as the equilibrium equations.

We conclude this section with a consideration of the dynamic force and moment vectors $\boldsymbol{p}^{(I)}$ and $\boldsymbol{q}^{(I)}$. From Eqs. (32) and (15), neglecting the warping of the cross section, we have

$$\boldsymbol{p}^{(I)} = p_i \boldsymbol{A}_i \qquad \boldsymbol{q}^{(I)} = q_i \boldsymbol{A}_i \tag{56}$$

$$\begin{aligned} p_1 &= \overline{\rho A}\ddot{u}_1 - \overline{\rho I_2}\ddot{\omega}_3 \\ p_2 &= \overline{\rho A}\ddot{u}_2 + \overline{\rho I_1}\ddot{\omega}_3 \\ p_3 &= \overline{\rho A}\ddot{u}_3 - \overline{\rho I_1}\ddot{\omega}_2 + \overline{\rho I_2}\ddot{\omega}_1 \end{aligned} \tag{57}$$

$$\begin{aligned} q_1 &= \overline{\rho I_2}\ddot{u}_3 + \overline{\rho I_{22}}\ddot{\omega}_1 - \overline{\rho I_{12}}\ddot{\omega}_2 \\ q_2 &= \overline{\rho I_1}\ddot{u}_3 + \overline{\rho I_{11}}\ddot{\omega}_2 - \overline{\rho I_{12}}\ddot{\omega}_1 \\ q_3 &= \overline{\rho I_1}\ddot{u}_2 - \overline{\rho I_2}\ddot{u}_1 + \overline{\rho I_{\alpha\alpha}}\ddot{\omega}_3 \end{aligned} \tag{58}$$

where

$$\begin{aligned} \overline{\rho A} &= \iint \rho \, d\theta_1 d\theta_2 \\ \overline{\rho I_\alpha} &= \iint \rho\theta_\alpha \, d\theta_1 d\theta_2 \\ \overline{\rho I_{\alpha\beta}} &= \iint \rho\theta_\alpha\theta_\beta \, d\theta_1 d\theta_2 \end{aligned} \tag{59}$$

VI. Untwisted Plane-Curved Rods

In the following we specialize in the rod equations given in the preceding sections for the case of a rod whose axis C forms an untwisted plane curve. We assume the geometry of the rod cross section, distribution of initial stresses, and the distribution of constituent materials to be symmetric about the plane of C. Taking A_1 to be on this plane, we have

$$\begin{aligned} &\mathcal{K}_1 = \mathcal{K}_3 = 0 \qquad \overset{(o)}{N}_2 = \overset{(o)}{M}_1 = \overset{(o)}{M}_3 = 0 \\ &\overset{(o)}{P}_{12} = \overset{(o)}{P}_{21} = \overset{(o)}{P}_{22} = 0 \qquad \overset{(o)}{M}_{12} = 0 \\ &\overline{EI_2} = \overline{EI_{12}} = \overline{EI}_{\alpha\alpha 2} = \theta_2^{(s)} = \theta_2^{(n)} = 0 \end{aligned} \tag{60}$$

The strain-displacement relations [Eq. (41)], the stress resultant and moment relations [Eq. (43)], and the constitutive relations [Eq. (44)] now take the form

$$\begin{aligned} &\gamma_1 = u_1' - \omega_2 + \mathcal{K}_2 u_3, \quad \gamma_2 = u_2' + \omega_1, \quad \epsilon = u_3' - \mathcal{K}_3\omega_1, \\ &\kappa_3 = \omega_3' - \mathcal{K}_2\omega_1, \quad \kappa_1 = \omega_1' + \mathcal{K}_2\omega_3, \quad \kappa_2 = \omega_2' \end{aligned} \tag{61}$$

$$\begin{aligned} &N_1 = N_1^* + \overset{(o)}{N}_3\omega_2, \quad M_1 = M_1^* \\ &N_2 = N_2^* - \overset{(o)}{N}_3\omega_1 + \overset{(o)}{N}_1\omega_3, \quad M_2 = M_2^* + \overset{(o)}{P}_{11}\omega_2 \\ &N_3 = N_3^* - \overset{(o)}{N}_1\omega_2, \quad M_3 = M_3^* + \overset{(o)}{M}_2\omega_1 + \overset{(o)}{P}_{11}\omega_3 \end{aligned} \tag{62}$$

$$\begin{aligned} &N_1^* = (\overline{GA} + \overset{(o)}{N}_3)\gamma_1, \quad N_2^* = (\overline{GA} + \overset{(o)}{N}_3)\gamma_2 - \overset{(o)}{M}_2\kappa_3 \\ &N_3^* = (\overline{EA} + \overset{(o)}{N}_3)\epsilon - (\overline{EA}\theta_1^{(n)} - \overset{(o)}{M}_2)\kappa_2 \\ &M_1^* = (EI_{22} + \overset{(o)}{M}_{22})\kappa_1, \\ &M_2^* = (\overline{EI_{11}} + \overset{(o)}{M}_{11})\kappa_2 - (EA\theta_1^{(n)} - \overset{(o)}{M}_2)\epsilon \\ &M_3^* = (\overline{GJ} + \overset{(o)}{M}_{\alpha\alpha})\kappa_3 - (\overline{GA}\theta_1^{(s)} + \overset{(o)}{M}_2)\gamma_2 \end{aligned} \tag{63}$$

The equilibrium relations (51) and (55) reduce to

$$N_1' + N_3\mathcal{K}_2 + p_1 = 0 \tag{64a}$$

$$N_2' + p_2 = 0 \tag{64b}$$

$$N_3' - N_1\mathcal{K}_2 + p_3 = 0 \tag{64c}$$

$$M_1' + M_3\mathcal{K}_2 - N_2 + \overset{(o)}{N}_3(\gamma_2 - \omega_1) + \overset{(o)}{N}_1\omega_3 - \overset{(o)}{M}_2\kappa_3 + q_1 = 0 \tag{65a}$$

$$M_2' + N_1 - \overset{(o)}{N}_3(\gamma_1 + \omega_2) + \overset{(o)}{N}_1\epsilon - \overset{(o)}{P}_{11}\kappa_2 + q_2 = 0 \tag{65b}$$

$$M_3' - M_1\mathcal{K}_2 - \overset{(o)}{N}_1(\gamma_2 - \omega_1) + \overset{(o)}{P}_{11}\kappa_3 + q_3 = 0 \tag{65c}$$

Use of Eqs. (61-63) in Eqs. (64a) and (64c) and in Eq. (65b) gives a system of three coupled differential equations for the in-plane deformation measures u_1, u_3, and ω_2, namely,

$$\begin{aligned} &\{\overline{GA} + \overset{(o)}{N}_3)(u_1' + \mathcal{K}_2u_3) - \overline{GA}\omega_2\}' + \mathcal{K}_2\{(\overline{EA} + \overset{(o)}{N}_3) \\ &\quad \times (u_3' - \mathcal{K}_2u_1) + \overset{(o)}{M}_2\omega_2' - \overset{(o)}{N}_1\omega_2\} + p_1 = 0 \\ &\{(\overline{EA} + \overset{(o)}{N}_3)(u_3' - \mathcal{K}_2u_1) + \overset{(o)}{M}_2\omega_2' - \overset{(o)}{N}_1\omega_2\}' \\ &\quad - \mathcal{K}_2\{(\overline{GA} + \overset{(o)}{N}_3)(u_1' + \mathcal{K}_2u_3) - \overline{GA}\omega_2\} + p_3 = 0 \\ &\{(\overline{EI}_{11} + \overset{(o)}{M}_{11})\omega_2' + \overset{(o)}{M}_2(u_3' - \mathcal{K}_2u_1)\}' + \overline{GA}(u_1' - \omega_2 + \mathcal{K}_2u_3) \\ &\quad + \overset{(o)}{N}_1(u_3' - \mathcal{K}_2u_1) + \overset{(o)}{P}_{11}'\omega_2 + q_2 = 0 \end{aligned} \tag{66}$$

where C has been chosen to coincide with the neutral axis ($\theta_\alpha^{(n)} = 0$).

Similarly, from Eqs. (64b), and (65a), and (65b), we obtain a system of equations governing the out-of-plane deformation measures u_2, ω_1, and ω_3, namely,

$$\begin{aligned} &\{(\overline{EI}_{22} + \overset{(o)}{M}_{22})(\omega_1' + \mathcal{K}_2\omega_3)\}' \\ &\quad + \mathcal{K}_2\{(\overline{GJ} + \overset{(o)}{M}_{\alpha\alpha})(\omega_3' - \mathcal{K}_2\omega_1) - \overset{(o)}{M}_2u_2' + P_{11}\omega_3\} \\ &\quad - \overline{GA}(u_2' + \omega_1) + q_1 = 0 \\ &\{(\overline{GJ} + \overset{(o)}{M}_{\alpha\alpha})(\omega_3' - \mathcal{K}_2\omega_1) - \overset{(o)}{M}_2u_2' + \overset{(o)}{P}_{11}\omega_3\}' \\ &\quad - \mathcal{K}_2(\overline{EI}_{22} + \overset{(o)}{M}_{22})(\omega_1' + \mathcal{K}_2\omega_3) - \overset{(o)}{N}_1u_2' \\ &\quad + \overset{(o)}{P}_{11}(\omega_3' - \mathcal{K}_2\omega_1) + q_3 = 0 \\ &\{\overline{GA}(u_2' + \omega_1) + \overset{(o)}{N}_3u_2' - \overset{(o)}{M}_2(\omega_3' - \mathcal{K}_2\omega_1) + \overset{(o)}{N}_1\omega_3\}' + p_2 = 0 \end{aligned} \tag{67}$$

where C has been chosen to coincide with the elastic axis ($\theta_\alpha^{(s)} = 0$).

We conclude with an examination of the systems of Eqs. (66) and (67) in the event that the transverse shear deformations are negligibly small. The limiting forms of these equations, when $\gamma_\alpha \to 0$, $\overline{GA}\,\gamma_\alpha \to$ finite values, are given by

$$
\begin{aligned}
&\omega_2 = u_1' + \mathcal{K}_2 u_3\\
&N_1' = -p_1 - \mathcal{K}_2\{(\overline{EA} + \overset{(o)}{N}_3)(u_3' - \mathcal{K}_2 u_1)\\
&\quad + \overset{(o)}{M}_2 (u_1' + \mathcal{K}_2 u_3)' - \overset{(o)}{N}_1 (u_1' + \mathcal{K}_2 u_3)\}\\
&\{\overline{EA} + \overset{(o)}{N}_3)(u_3' - \mathcal{K}_2 u_1) + \overset{(o)}{M}_2 (u_1' + \mathcal{K}_2 u_3)'\\
&\quad - \overset{(o)}{N}_1 (u_1' + \mathcal{K}_2 u_3)\}' - \mathcal{K}_2 N_1 + p_3 = 0\\
&\{(\overline{EI}_{11} + \overset{(o)}{M}_{11})(u_1' + \mathcal{K}_2 u_3)' + \overset{(o)}{M}_2 (u_3' - \mathcal{K}_2 u_1)\}'\\
&\quad + N_1 - (\overset{(o)}{N}_3 - \overset{(o)}{P}_{11})(u_1' + \mathcal{K}_2 u_3) + \overset{(o)}{N}_1 (u_3' - \mathcal{K}_2 u_1) + q_2 = 0
\end{aligned}
\tag{68}
$$

and

$$
\begin{aligned}
&\omega_1 = -u_2'\\
&N_2' = -p_2\\
&\{(\overline{EI}_{22} + \overset{(o)}{M}_{22})(\omega_1' + \mathcal{K}_2 \omega_3)\}' + \mathcal{K}_2\{(\overline{GJ} + \overset{(o)}{M}_{\alpha\alpha})(\omega_3' - \mathcal{K}_2 \omega_1)\\
&\quad + \overset{(o)}{M}_2 \omega_1 + P_{11}\omega_3\} - N_2 - \overset{(o)}{N}_3 \omega_1 - \overset{(o)}{M}_2 (\omega_3' - \mathcal{K}_2 \omega_1)\\
&\quad + \overset{(o)}{N}_1 \omega_3 + q_1 = 0\\
&\{(\overline{GJ} + \overset{(o)}{M}_{\alpha\alpha})(\omega_3' - \mathcal{K}_2 \omega_1) + \overset{(o)}{M}_2 \omega_1 + \overset{(o)}{P}_{11}\omega_3\}'\\
&\quad - \mathcal{K}_2 (\overline{EI}_{22} + \overset{(o)}{M}_{22})(\omega_1 + \mathcal{K}_2 \omega_3)\\
&\quad + \overset{(o)}{N}_1 \omega_1 + \overset{(o)}{P}_{11}(\omega_3' - \mathcal{K}_2 \omega_1) + q_3 = 0
\end{aligned}
\tag{69}
$$

In the case of straight, initially unstressed rods, Eqs. (68) and (69) further reduce to the equations governing extension, bending, and torsion in accordance with elementary theories.

VII. Remarks

In the foregoing, a three-dimensional structural model for initially stressed, curved, twisted rods is developed. The theory describes infinitesimal incremental displacements and associated stresses using the principle of virtual work. Although the initial configuration of the rod is assumed to be static, the equations obtained here are suitable for the treatment of superimposed, dynamic disturbances. The variational formulation has the advantage of a symmetric system of constitutive equations. These equations show that the stress resultants and couples may be decomposed into two parts—one depending on the initial stress resultants and the infinitesimal rotations; the other depending on the incremental strain measures and the material properties of the rod. Apart from the well-known[1,2] *effective torsional rigidity* in the presence of axial stress, namely, the St. Venant torsional rigidity $\overline{GJ}$ modified by a term containing $\iint \overset{(o)}{\sigma}{}^{33}(\theta_1^2 + \theta_2^2)\,d\theta_1 d\theta_2$, we observe that our constitutive relations exhibit modified shear rigidities as well as bending rigidities.

The general system equations are also presented in a specialized form for the treatment of plane-curved rods. These equations have the remarkable property that they are uncoupled as far as the in-plane and the out-of-plane deformations are concerned. The consideration of the shear center and the inclusion of shear deformation effects are additional features of the present derivation.

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